Midterm Answers

1. We need to substitute for Q in the profit function so that profit depends on the single decision variable P. Substituting Q = a - bP gives

$$\begin{aligned}\pi &= PQ - e - fQ \\ &= P(a - bP) - e - f(a - bP) \\ &= aP - bP^2 - e - af + bfP\end{aligned}$$

   a. The derivative of π with respect to P is

$$\frac{d\pi}{dP} = a - 2bP + bf$$

   Setting this equal to zero:

$$\begin{aligned}P &= \frac{a + bf}{2b} \\ &= 0.5\left(\frac{a}{b} + f\right)\end{aligned}$$

   b. This is a minimum since the second derivative is negative:

$$\frac{d^2\pi}{dP^2} = -2b < 0$$

   c. An increase in b reduces the value of the profit-maximizing price:

$$\frac{dP}{db} = -0.5\frac{a}{b^2}$$

   d. It makes sense that an increase in b, indicating demand is more sensitive to price, reduces the monopolist's profit-maximizing price. With a higher value of b, an increase in P reduces demand by more.

2. If T is the tax rate, then tax revenue is R = TPQ. and we know that the percentage change in R is approximately given by this equation: %R = %T + %P +%Q. Therefore, %T = %R - %P - %Q = 3% - 3% - 3% = -3%. The tax rate fell by 3%.

3. The objective function is

$$Z = 10 + 8Q_1 + 1Q_1^2 + 10 + 2Q_2 + 2Q_2^2 + \lambda(12 - Q_1 - Q_2)$$

   a. The first derivatives are

$$\frac{\partial Z}{\partial Q_1} = 8 + 2Q_1 - \lambda$$

$$\frac{\partial Z}{\partial Q_2} = 2 + 4Q_2 - \lambda$$

$$\frac{\partial Z}{\partial \lambda} = -Q_1 - Q_2 + 12$$

   The first derivatives equal zero if 8 + $2Q_1$ = 2 + $4Q_2$. Using the total production constraint, $Q_1 + Q_2 = 12$,we can solve for the optimal quantities, $Q_1 = 7$ and $Q_2 = 5$:

$$\begin{aligned} 8 + 2Q_1 &= 2 + 4Q_2 \\ &= 2 + 4(12 - Q_1) \\ &= 50 - 4Q_1 \\ 6Q_1 &= 42 \\ Q_1 &= 7 \end{aligned}$$

b. $\dfrac{dC_1}{dQ_1} = 8 + 2Q_1 = 8 + 2(7) = 22$

c. $\dfrac{dC_2}{dQ_2} = 2 + 4Q_2 = 2 + 4(5) = 22$

d. The first-order condition in Part (a), $8 + 2Q_1 = 2 + 4Q_2$, is a requirement that the marginal costs of production be equal in the two locations. This makes sense because if the marginal cost were higher in one location than in the other, the firm could reduce its costs by producing 1 less in the high-cost location and 1 more in the low-cost location.

4. The first and second derivatives of Z with respect to n are

$$\frac{dZ}{dn} = \frac{2nRY - 2(n-1)RY}{4n^2} - C = \frac{RY}{2n^2} - C$$

$$\frac{d^2Z}{dn^2} = -\frac{RY}{n^3} < 0$$

This is a maximum since the second derivative is negative.

a. The first derivative equals 0 at

$$C = \frac{RY}{2n^2}$$

$$n^2 = \frac{RY}{2C}$$

$$n = \sqrt{\frac{RY}{2C}}$$

b. For the given parameter values,

$$n = \sqrt{\frac{.05(1000)}{2}} = 5$$

Average money holdings M are consequently given by this square-root rule:

$$M = \frac{Y}{2n} = \sqrt{\frac{CY}{2R}}$$

with money demand related positively to Y and negatively to R. The seminal papers are James Tobin, “The Interest-Elasticity of Transactions Demand for Cash,” Review of Economics and Statistics, 1956, 241-247; William Baumol, “The Transactions Demand for Cash: An Inventory Theoretic Approach,” Quarterly Journal of Economics, 1952, 545-556.

5. a. Because the capital is purchased today and yields profits in the future, we need to calculate the present value of the cash flow:

$$V = -P_K K + \frac{PQ - wL}{(1+R)^1} + \frac{PQ - wL}{(1+R)^2} + \ldots$$

where R is the firm’s required rate of return. Using the consol formula

$$V = -P_K K + \frac{PQ - wL}{R}$$

The first-order conditions are

$$0 = \frac{\partial V}{\partial K} = -P_K + \frac{P\frac{\partial Q}{\partial K}}{R}$$

$$0 = \frac{\partial V}{\partial L} = \frac{P\frac{\partial Q}{\partial L} - w}{R}$$

Thus

$$\frac{\partial Q}{\partial K} = \frac{RP_K}{P}$$

$$\frac{\partial Q}{\partial L} = \frac{w}{P}$$

b. The ratio of the marginal product of capital to the marginal product of labor is

$$\frac{\partial Q / \partial K}{\partial Q / \partial L} = \frac{RP_K}{w}$$

The numerator is the annual cost of a unit of capital: the required return times the price of capital. For example, if a unit of capital cost P_K = \$1,000 and the required return is $R = 0.10$, then it costs \$100/year to employ capital. Thus, the ratio of the marginal product of capital to the marginal product of labor is equal to the ratio of the cost of capital to the cost of labor.

[not asked]. If we allow for depreciation expenses:

$$V = -P_K K + \frac{PQ - wL - \delta P_K K}{(1+R)^1} + \frac{PQ - wL - \delta P_K K}{(1+R)^2} + \dots$$

$$= -P_K K + \frac{PQ - wL - \delta P_K K}{R}$$

The first-order conditions are

$$0 = \frac{\partial V}{\partial K} = -P_K + \frac{P\frac{\partial Q}{\partial K} - \delta P_K K}{R}$$

$$0 = \frac{\partial V}{\partial L} = \frac{P\frac{\partial Q}{\partial L} - w}{R}$$

Thus

$$\frac{\partial Q}{\partial K} = \frac{P_K (R + \delta)}{P}$$

$$\frac{\partial Q}{\partial L} = \frac{w}{P}$$

[not asked]. If we let L and Q vary every period,

$$V = -P_K K + \frac{PQ_1 - wL_1}{(1+R)^1} + \frac{PQ_2 - wL_2}{(1+R)^2} + \dots$$

The first-order conditions show that the firm should hire the same amount of labor and produce the same output each period if the marginal product of labor is constant:

$$0 = \frac{\partial V}{\partial K} = -P_K + \frac{P\frac{\partial Q_1}{\partial K}}{(1+R)^1} + \frac{P\frac{\partial Q_2}{\partial K}}{(1+R)^2} + \dots$$

$$0 = \frac{\partial V}{\partial L_1} = \frac{P\frac{\partial Q_1}{\partial L_1} - w}{(1+R)^1}$$

$$0 = \frac{\partial V}{\partial L_2} = \frac{P\frac{\partial Q_2}{\partial L_2} - w}{(1+R)^1}$$