Econ 58
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Final Exam Answers

1. a. The exact change in nominal wages between 2003 and 2004:

$$\frac{W_{2004} - W_{2003}}{W_{2003}} = \frac{\$18.09 - \$17.75}{\$17.75}$$
$$= 0.01915 \ (1.915\%)$$

b. The exact change in real wages between 2003 and 2004:

$$\frac{\frac{W_{2004}}{P_{2004}} - \frac{W_{2003}}{P_{2003}}}{\frac{W_{2003}}{P_{2003}}} = \frac{\frac{\$18.09}{189.4} - \frac{\$17.75}{183.9}}{\frac{\$17.75}{183.9}}$$
$$= -0.0104 \ (-1.04\%)$$

c. The exact real interest rate in 2003:

$$\frac{1 + R_{2003}}{1 + \frac{P_{2004} - P_{2003}}{P_{2003}}} - 1 = \frac{1 + 0.010}{1 + \frac{189.4 - 183.9}{183.9}} - 1$$
$$= -0.01933 \ (-1.933\%)$$

The approximate real interest rate:

$$R_{2003} - \frac{P_{2004} - P_{2003}}{P_{2003}} = 0.010 - \frac{189.4 - 183.9}{183.9}$$
$$= -0.0199 \ (-1.99\%)$$

2. a. The slope is negative:

$$Y = \left[C - X^2\right]^{1/2}$$
$$\frac{dY}{dX} = \frac{1}{2}\left[C - X^2\right]^{-1/2}(-2X)$$
$$= -\frac{X}{Y}$$

This negative slope can also be determined this way:

$$X^2 + Y^2 = C$$
$$2X + 2Y\frac{dY}{dX} = 0$$
$$\frac{dY}{dX} = -\frac{X}{Y}$$

b. The slope is decreasing (becoming increasingly negative) since the second derivative is negative:

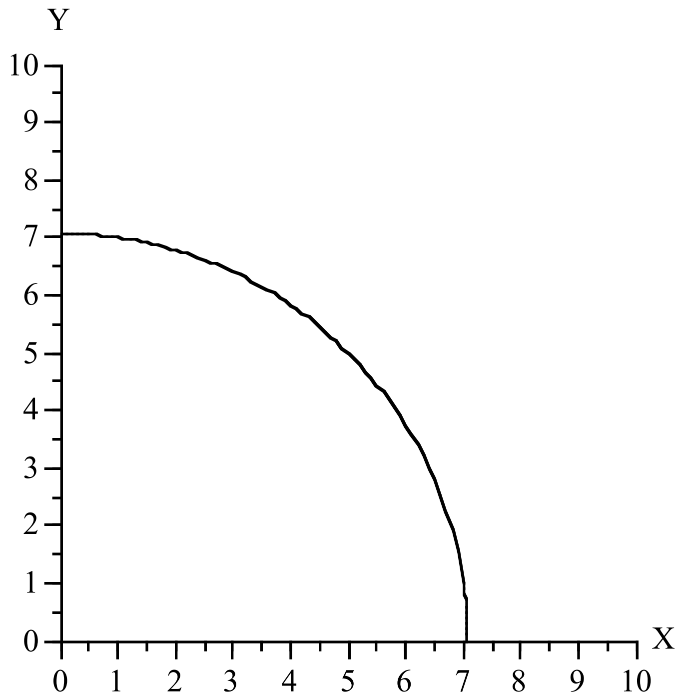
$$\frac{dY}{dX} = -X\left(C - X^2\right)^{-1/2}$$

$$\frac{d^2Y}{dX^2} = -\left(C - X^2\right)^{-1/2} - X\left(-1/2\right)\left(C - X^2\right)^{-3/2}\left(-2X\right)$$

$$= -\left(C - X^2\right)^{-1/2} - X^2\left(C - X^2\right)^{-3/2}$$

$$= -\frac{1}{Y} - \frac{X^2}{Y^3}$$

c. Here is a graph for C = 50:



3. a. The slope is negative:

$$U_0 = AX^{\alpha}Y^{\beta}$$

$$Y = \left(\frac{U_0}{A}\right)^{1/\beta} X^{-\alpha/\beta}$$

$$\frac{dY}{dX} = -\frac{\alpha}{\beta}\left(\frac{U_0}{A}\right)^{1/\beta} X^{-(\alpha/\beta)-1}$$

$$= -\frac{\alpha}{\beta}\frac{Y}{X} < 0$$

This negative slope can also be determined this way:

$$U_0 = AX^{\alpha}Y^{\beta}$$

$$0 = \alpha\frac{U_0}{X} + \beta\frac{U_0}{Y}\frac{dY}{dX} = 0$$

$$\frac{dY}{dX} = -\frac{\alpha}{\beta}\frac{Y}{X}$$

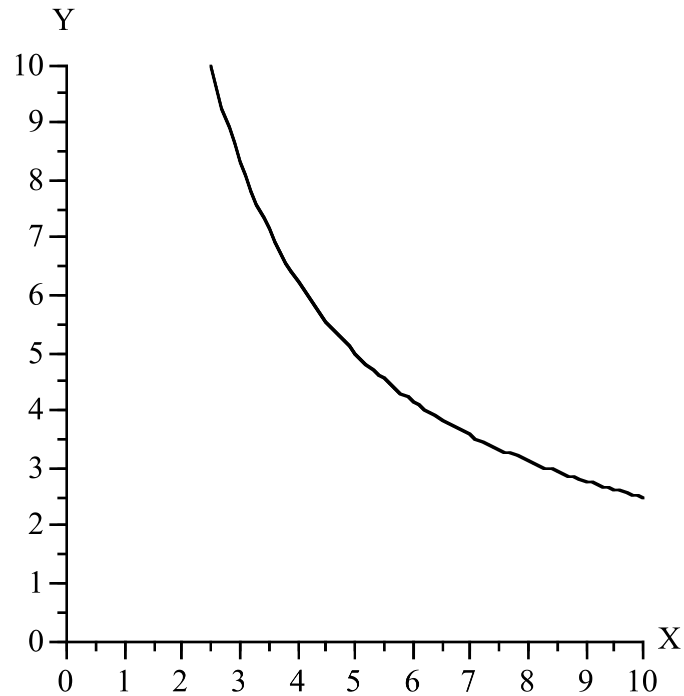
b. The slope is increasing (becoming less negative) since the second derivative is positive:

$$\frac{dY}{dX} = -\frac{\alpha}{\beta}\left(\frac{U_0}{A}\right)^{1/\beta} X^{-(\alpha/\beta)-1}$$

$$= -\frac{\alpha}{\beta}\left(\frac{U_0}{A}\right)^{1/\beta} X^{-(\beta+\alpha/\beta)}$$

$$\frac{d^2Y}{dX^2} = \frac{\alpha+\beta}{\beta}\frac{\alpha}{\beta}\left(\frac{U_0}{A}\right)^{1/\beta} X^{-(\alpha/\beta)-2} > 0$$

c. Here is a graph for $\alpha = \beta = 0.5$:



4. a. The objective function is

$$Z = AX^{\alpha}Y^{\beta} + \lambda\left[C - X^2 - Y^2\right]$$

The first-order conditions are

$$0 = \frac{\partial Z}{\partial X} = \alpha AX^{\alpha-1}Y^{\beta} - 2\lambda X$$

$$0 = \frac{\partial Z}{\partial Y} = \beta AX^{\alpha}Y^{\beta-1} - 2\lambda Y$$

$$\frac{\alpha Y}{\beta X} = \frac{X}{Y}$$

$$\frac{Y^2}{X^2} = \frac{\beta}{\alpha}$$

$$\frac{Y}{X} = \sqrt{\frac{\beta}{\alpha}}$$

Not asked: Using the PPF, we have

$$X^2 + Y^2 = C$$

$$X^2 + \frac{\beta}{\alpha}X^2 = C$$

$$\frac{\alpha + \beta}{\alpha}X^2 = C$$

$$X = \sqrt{\frac{\alpha C}{\alpha + \beta}}$$

$$Y^2 = C - X^2$$

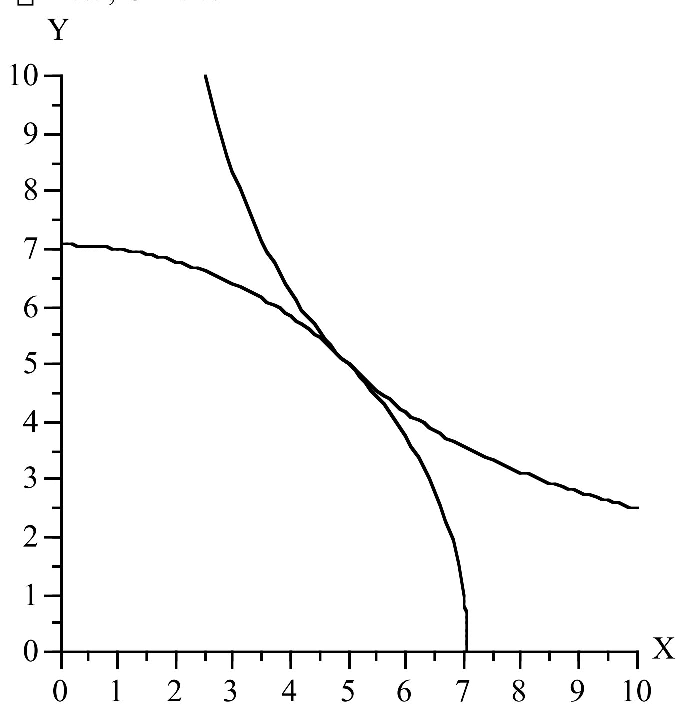
$$Y = \sqrt{\frac{\beta C}{\alpha + \beta}}$$

In the special case $\alpha = \beta = 0.5$, $C = 50$,

$$X = \sqrt{\frac{\alpha C}{\alpha + \beta}} = \sqrt{0.5(50)} = 5$$

$$Y = \sqrt{\frac{\beta C}{\alpha + \beta}} = \sqrt{0.5(50)} = 5$$

b. Here is a graph for $\alpha = \beta = 0.5$, $C = 50$:



5. The objective function is

$$Z = AX_C^{\alpha}Y_C^{\beta} + \lambda_1\left[C - X_C^2 - Y_C^2\right] + \lambda_2\left(P_X X + P_Y Y - P_X X_C - P_Y Y_C\right)$$

The first-order conditions are

$$0 = \frac{\partial Z}{\partial X_C} = \alpha A X_C^{\alpha-1} Y_C^{\beta} - \lambda_2 P_X$$

$$0 = \frac{\partial Z}{\partial Y_C} = \beta A X_C^{\alpha} Y_C^{\beta-1} - \lambda_2 P_Y$$

$$0 = \frac{\partial Z}{\partial X} = -2\lambda_1 X + \lambda_2 P_X$$

$$0 = \frac{\partial Z}{\partial Y} = -2\lambda_1 Y + \lambda_2 P_Y$$

Solving,

$$\alpha A X_C^{\alpha-1} Y_C^{\beta} = \lambda_2 P_X$$

$$\beta A X_C^{\alpha} Y_C^{\beta-1} = \lambda_2 P_Y$$

$$2\lambda_1 X = \lambda_2 P_X$$

$$2\lambda_1 Y = \lambda_2 P_Y$$

Taking ratios:

$$\frac{\alpha A X_C^{\alpha-1} Y_C^{\beta}}{\beta A X_C^{\alpha} Y_C^{\beta-1}} = \frac{\lambda_2 P_X}{\lambda_2 P_Y} \Rightarrow \frac{Y_C}{X_C} = \frac{\beta}{\alpha}\frac{P_X}{P_Y}$$

$$\frac{2\lambda_1 X}{2\lambda_1 Y} = \frac{\lambda_2 P_X}{\lambda_2 P_Y} \Rightarrow \frac{X}{Y} = \frac{P_X}{P_Y}$$

The price ratio is the terms at which the country can trade one commodity for another. The first condition states that the slope of the indifference curve is equal to this terms of trade (i.e., the terms at which the country is willing to trade one commodity for another is equal to the terms at which it can trade); the second condition states that the slope of the production function is equal to this terms of trade (i.e., the terms at which the country can trade one commodity for another in its domestic production is equal to the terms at which it can trade internationally.)

Not asked: In the special case a = b = 0.5, C = 50, $P_X/P_Y = 0.5$:

$$\frac{\alpha Y_C}{\beta X_C} = \frac{P_X}{P_Y} \Rightarrow \frac{0.5 Y_C}{0.5 X_C} = 0.5 \Rightarrow \frac{Y_C}{X_C} = 0.5$$

$$\frac{X}{Y} = \frac{P_X}{P_Y} = 0.5$$

Using the PPF, we have

$$X^2 + Y^2 = C$$

$$(0.5Y)^2 + Y^2 = 50$$

$$1.25Y^2 = 50$$

$$Y = \sqrt{\frac{50}{1.25}}$$

$$= \sqrt{40}$$

$$X^2 = 50 - Y^2$$

$$= 50 - 40$$

$$X = \sqrt{10}$$

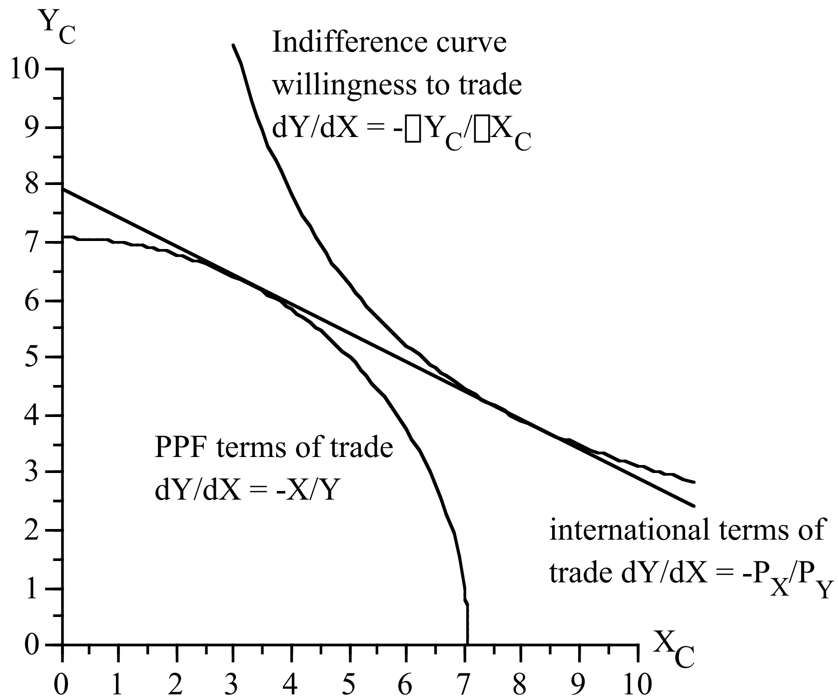
Using the budget constraint,

$$P_x X_c + P_y Y_c = P_x X + P_y Y$$

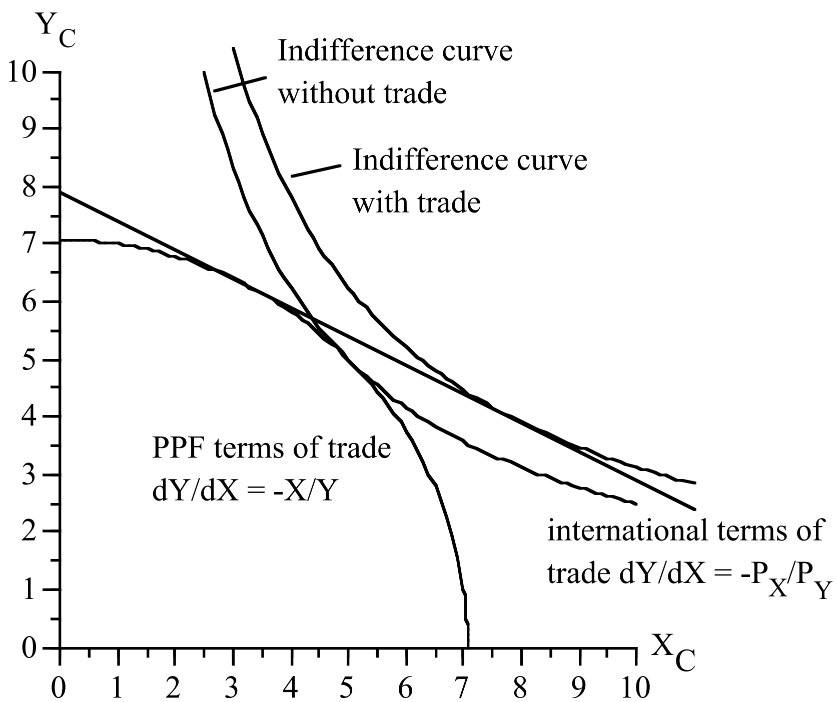
$$X_c + 2(0.5X_c) = \sqrt{10} + 2\sqrt{40}$$

$$X_c = \frac{\sqrt{10} + 2\sqrt{40}}{2}$$

Here is a graph for □ = □ = 0.5, C = 50, $P_X/P_Y = 0.5$:



This analysis shows how international trade can improve a country's well-being:



6. a. Using a continuous discount rate, the net present value is

$$V = Ce^{\alpha t - \beta t^2} e^{-Rt} - C$$
$$= Ce^{\alpha t - \beta t^2 - Rt} - C$$

The first derivative is

$$\frac{dV}{dt} = \left(\alpha - 2\beta t - R\right)Ce^{\alpha t - \beta t^2 - Rt}$$

which equals 0 when

$$\alpha - 2\beta t - R = 0$$
$$t = \frac{\alpha - R}{2\beta}$$

b. The percentage increase in the value of the fish is

$$\frac{dP}{dt}\frac{1}{P} = \frac{\left(\alpha - 2\beta t\right)Ce^{\alpha t - \beta t^2}}{Ce^{\alpha t - \beta t^2}}$$
$$= \alpha - 2\beta t$$

which is shown above to equal R. This is because the optimal time to harvest fish is when the rate of increase in the value of the fish is equal to the rate of return one could earn if the fish were harvested and the proceeds invested at a rate of return R. If the percentage increase in the value of the fish were greater than R, it would be more profitable to let the fish keep increasing in value; if the percentage increase in the value of the fish were less than R, it would have been more profitable to harvest the fish sooner.

7. a. There is two noncooperative equilibria, at low, high and high, low.

b. The expected values of the payoffs are

$$E_1 = 50pq + 0p(1-q) + 75(1-p)q - 25(1-p)(1-q)$$
$$E_2 = 50pq + 75p(1-q) + 0(1-p)q - 25(1-p)(1-q)$$

The first-order conditions are

$$\frac{dE_1}{dp} = 50q - 75q + 25(1-q) \Rightarrow \quad q = \frac{25}{50} = 1/2$$
$$\frac{dE_2}{dq} = 50p - 75p + 25(1-p) \Rightarrow \quad p = \frac{25}{50} = 1/2$$

8. Substituting:

$$Y = c_0 + c_y(Y_{-1} - T_{-1}) + i_0 + g_0$$
$$= c_0 + c_y(Y_{-1} - t_y Y_{-1}) + i_0 + g_0$$
$$= c_0 + i_0 + g_0 + c_y(1 - t_y)Y_{-1}$$

In dynamic equilibrium,

$$Y^e = c_0 + i_0 + g_0 + c_y(1 - t_y)Y^e$$

Subtracting the second equation from the first

$$Y = c_0 + i_0 + g_0 + c_y(1 - t_y)Y_{-1}$$
$$Y^e = c_0 + i_0 + g_0 + c_y(1 - t_y)Y^e$$
$$Y - Y^e = c_y(1 - t_y)\left[Y_{-1} - Y^e\right]$$

This is stable if and only if $\left|c_y(1 - t_y)\right| < 1$. Because all the terms are positive, stability requires $c_y(1 - t_y) < 1$.

Thus stability is more likely to be stable if t_y is large. Logically, an increase in income raises taxes, which dampens spending and makes it less likely that the model will be monotonically unstable. (The model is definitely monotonically stable if $c_y < 1$.

9. Writing these two equations in matrix form

$$\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} S_1 - d_{10} \\ S_2 - d_{20} \end{bmatrix}$$

There is unique solution if the determinant is not equal to zero: $d_{11}d_{22} - d_{12}d_{21} \neq 0$

10. The central equation should be something like this:

$$E_t = E_{t-1}(1 + R_t) + D_t - W_t$$

where E is the real value of the endowment, R is the real rate of return, D is real donations, and W is real withdrawals. We might assume that the real values of donations and withdrawals grow over time (or are constant with g = 0):

$$D_t = (1 + g_D)D_{t-1} + \varepsilon_t$$

$$W_t = (1 + g_W)W_{t-1} + \nu_t$$

The error terms allow donations and withdrawals to fluctuate year-to-year about their long-run growth paths. We can use historical data to estimate the donation and withdrawal growth rates and also probability distributions for the error terms and for R. Run a large number of Monte Carlo simulations with the rate of return, donation error term, and withdrawal error term detemined by a random number generator and, for each simulation, record the value of the endowment after 50 years and whether the endowment ever dropped below $0.5E_0$. These statistics can be used to estimate the median value of the endowment after 50 years and the probability that it will fall below $0.5E_0$ at some point.