Econ 58
Gary Smith
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Midterm Answers

1. a. In the Cobb-Douglas production function, $Q = AK^{\alpha}L^{\beta}$, the elasticity of output with respect to capital is a and the elasticity of output with respect to labor is b. Here, with a = 0.4, a 1% increase in K increases Q by 0.4% and a 5% increase in K increases Q by 5(0.4) = 2.0%.

   b. The percentage change in X/Y is approximately equal to the percentage change in X minus the percentage change in Y. Here, if K increases by 3% and L increases by 4%, the approximate percentage change in K/L is 3% - 4% = -1%.

   c. The percentage change in XY is approximately equal to the percentage change in X plus the percentage change in Y. Here, if P increases by 5% and Q falls by 4%, the approximate percentage change in PQ is 5% + (-4%) = 1%.

   d. The percentage change in Q[K, L] is approximately equal to the elasticity of Q with respect to K multiplied by the percentage change in K, plus the elasticity of Q with respect to L multiplied by the percentage change in L. Here, as in part a, the elasticities are 0.4 and 0.6 and the approximate percentage increase in Q is 0.4(5%) + 0.6(5%) = 5%. More generally if the elasticities add to 1, then the Cobb-Douglas production function exhibits constant returns to scale in that an x% change in all inputs will increase output by x%.

2. a. Taking the partical derivative:

$$\frac{\partial V}{\partial R} = \frac{-D}{(R-g)^2}$$

Thus the elasticity is

$$\epsilon = \left|\frac{\partial V}{\partial R}\frac{R}{V}\right|$$
$$= \left|\left(\frac{-D}{(R-g)^2}\right)\left(\frac{R}{D/(R-g)}\right)\right|$$
$$= \frac{R}{R-g}$$

   b. The partical derivative of the elasticity with respect to g is positive

$$\frac{\partial \epsilon}{\partial g} = \frac{R}{(R-g)^2} > 0$$

Thus the value of a growth stock is more sensitive to changes in R.

3.a. Holding Y constant, the first derivative of utility with respect to X is

$$\frac{\partial U}{\partial X} = \frac{3}{X} > 0 \text{ (positive marginal utility)}$$

   b. Holding Y constant, the second derivative of utility with respect to X is

$$\frac{\partial^2 U}{\partial X^2} = -\frac{3}{X^2} < 0 \text{ (diminishing marginal utility)}$$

   c. Taking the total derivative, we find that the slope of the indifference curve is negative:

$$U_0 = 3\ln[X] + \ln[Y]$$
$$dU_0 = \frac{\partial U}{\partial X}dX + \frac{\partial U}{\partial Y}dY$$
$$0 = \frac{3}{X}dX + \frac{1}{Y}dY$$
$$\frac{dY}{dX} = -3\frac{Y}{X} = -3 \text{ at } X = Y$$

d. The negative slope of the indifference curve means that if this person obtains more X, utility will not be constant unless this person has less Y. The first derivative gives the terms at which this person is willing to trade Y for X. For example, at X = Y, dY/dX = -3 means that this person is willing to trade 3 units of Y for 1 unit of X.

4. a. The profit function is

$$\pi = PQ - C$$
$$= \left(7500Q^{-0.5}\right)Q - 100Q^{1.5}$$
$$= 7500Q^{0.5} - 100Q^{1.5}$$

The first and second derivatives are

$$\pi = PQ - C$$
$$= 7500Q^{0.5} - 100Q^{1.5}$$
$$\frac{d\pi}{dQ} = 3750Q^{-0.5} - 150Q^{0.5} = 0 \text{ at } Q = 25$$
$$\frac{d^2\pi}{dQ^2} = -1875Q^{-1.5} - 75Q^{-0.5} < 0$$

The negative second derivative tells us that Q = 25 is a maximum.

b. Marginal cost is

$$\frac{dC}{dQ} = 150Q^{0.5} = 750$$

c. Marginal revenue is:

$$\frac{d(PQ)}{dQ} = 3750Q^{-0.5} = 750 \text{ at } Q = 25$$

d. Marginal revenue is equal to marginal cost at the profit-maximizing output, because the level of output can only be profit-maximizing if producing more does not increase revenue more than it increases cost and if producing less does not decrease revenue less than it decreases cost; that is, only if marginal revenue is equal to marginal cost.

5. a. The net present value is

$$P[t] = V[t]e^{-Rt} - C$$
$$= 100t^{0.9}e^{-0.05t} - 50$$

b. We can set the first derivative equal to zero:

$$\frac{dP[t]}{dt} = 100t^{0.9}e^{-0.05t}(-0.05) + 0.9(100)t^{-0.1}e^{-0.05t}$$

$$= -5t^{0.9}e^{-0.05t} + 90t^{-0.1}e^{-0.05t}$$

$$= e^{-0.05t}\left[90t^{-0.1} - 5t^{0.9}\right]$$

$$= 0 \text{ when}$$

$$t = \frac{90}{5} = 18$$

c. The second derivative is negative (indicating a maximum):

$$\frac{dP[t]}{dt} = e^{-0.05t}\left[90t^{-0.1} - 5t^{0.9}\right]$$

$$\frac{d^2P[t]}{dt^2} = -0.05e^{-0.05t}\left[90t^{-0.1} - 5t^{0.9}\right] + e^{-0.05t}\left[-9t^{-1.1} - 4.5t^{-0.1}\right]$$

$$< 0 \text{ at } \frac{dP[t]}{dt} = 0$$

d. In general, the differentiation of P[t] with respect to t shows that W'[t] = 0 when V'[t]/V[t] = R, that is, the rate of change of the harvest value is equal to the real interest rate:

$$P[t] = V[t]e^{-Rt} - C$$

$$\frac{dP[t]}{dt} = V[t]e^{-Rt}(-R) + V'[t]e^{-Rt}$$

$$= 0 \text{ when}$$

$$\frac{V'[t]}{V[t]} = R$$

Thus the trees should be kept growing as long as the increase in the value of the stand is larger than the interest rate.