Econ 58
Gary Smith
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Midterm Answers

1. a. Holding Y constant, the first derivative of utility with respect to X is

$$\frac{\partial U}{\partial X} = 0.5AX^{-0.5}Y^{0.5} > 0 \text{ (positive marginal utility)}$$

b. Holding Y constant, the second derivative of utility with respect to X is

$$\frac{\partial^2 U}{\partial X^2} = -0.25AX^{-1.5}Y^{0.5} < 0 \text{ (diminishing marginal utility)}$$

c. Solving for Y as a function of X:

$$U_0 = A\sqrt{XY}$$
$$U_0^2 = A^2XY$$
$$Y = \frac{U_0^2}{A^2X}$$

d. The slope of the indifference curve is negative:

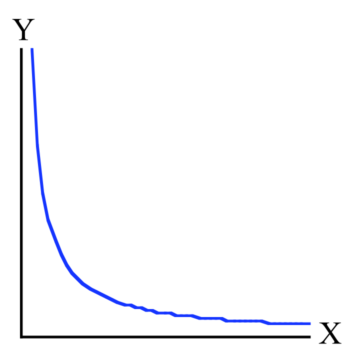
$$\frac{dY}{dX} = -\frac{U_0^2}{A^2X^2}$$
$$= -\frac{A^2XY}{A^2X^2}$$
$$= -\frac{Y}{X} < 0$$

e. The second derivative is positive:

$$\frac{d^2Y}{dX^2} = 2\frac{U_0^2}{A^2X^3} = 2\frac{Y}{X^2} > 0$$

Many plausible indifference curves have a negative first derivative and positive second derivative. The negative slope means that if this person obtains more X, utility will not be constant unless this person has less Y. The first derivative gives the terms at which this person is willing to trade Y for X. The positive second derivative means that as this person obtains more X, the amount of Y the person is willing to give up to get even more X diminishes towards zero. As Y becomes infinitely large, the slope approaches 0.

f. Here is a sketch:



g. The negative slope of the indifference curve means that if this person obtains more X, utility will not be constant unless this person has less Y. The first derivative gives the terms at which this person is willing to trade Y for X. For example, at X = Y, dY/dX = -1 means that this person is willing to trade 3 units of Y for 1 unit of X.

h. Because this person is willing to trade 1 unit of Y for 1 unit of X, and the prices allow him/her to obtain 2 unit of X for only one unit of Y, he/she will gladly trade Y to obtain more X.

2. a. This is the elasticity of output with respect to capital.
   b. The objective function is

$$Z = rK + wL + \lambda\left[100 - AK^{\alpha}L^{\beta}\right]$$

   The first derivatives are

$$\frac{\partial Z}{\partial K} = r - \lambda\left[\alpha AK^{\alpha-1}L^{\beta}\right] = r - \lambda\alpha\frac{Q}{K}$$

$$\frac{\partial Z}{\partial L} = w - \lambda\left[\beta AK^{\alpha}L^{\beta-1}\right] = w - \lambda\beta\frac{Q}{L}$$

$$\frac{\partial Z}{\partial \lambda} = -AK^{\alpha}L^{\beta} + 100$$

   Equating the first derivatives to 0 gives

$$\frac{\alpha\frac{Q}{K}}{\beta\frac{Q}{L}} = \frac{r}{w}$$

   so that the optimal capital-labor ratio is

$$\frac{K}{L} = \frac{\alpha w}{\beta r}$$

   c. The lefthand side of the final equation in part (b) is the ratio of the marginal products. Thus the ratio of the marginal products is equal to r/w, which means the ratio of the marginal product of capital to the marginal product of labor equals the ratio of the marginal cost of capital to the marginal cost of labor.
   d. For the values given, the ratio of the marginal products is 1/3 and the ratio of the marginal costs is 0.1/0.2 = 1/2. They should substitute labor for capital as explained in the answer to Part (e).
   e. The marginal product of labor is 3 times the marginal product of capital, but the marginal cost of labor is only twice the marginal cost of capital. Therefore, they can reduce costs by replacing capital with labor.

3. a. Substitute the demand function and cost functions into the profit function:

$$\pi = \left(100 - 2(Q_1 + Q_2)\right)(Q_1 + Q_2) - \left[20 + Q_1^2 + 30 + Q_2^2\right]$$

$$= 100Q_1 + 100Q_2 - 2Q_1^2 - 4Q_1Q_2 - 2Q_2^2 - 20 - Q_1^2 - 30 - Q_2^2$$

   The first derivatives are

$$\frac{\partial \pi}{\partial Q_1} = 100 - 6Q_1 - 4Q_2$$

$$\frac{\partial \pi}{\partial Q_2} = 100 - 6Q_2 - 4Q_1$$

   Setting the first derivatives equal to zero, the solutions are $Q_1 = Q_2 = 10$.
   b. The fact that the locations have different fixed costs, 20 and 30, is irrelevant, since the firm's production decisions are based on marginal costs and marginal revenues, and these products have the same marginal cost equation and also the same marginal revenue equation.

4. a. The exogenous variables g_0 is the value of g at time t = 0.
   b. The parameter $\square$ is the continuously compounded rate of growth of government spending.
   c. Substituting for g and h, the equation for z/y is

$$\frac{z}{y} = \frac{g_0 e^{\square t} \square \square y}{y}$$
$$= \frac{g_0 e^{\square t}}{y} \square \square$$
$$= \frac{g_0 e^{\square t}}{y_0 e^{\square t}} \square \square$$
$$= \frac{g_0}{y_0} e^{(\square \square \square)t} \square \square$$

   d. The deficit-income ratio z/y will fall over time if $\square - \square < 0$; i.e., $\square$ is less than $\square$.
   e. Government spending must grow more slowly than national income.

5. a. We can use the Lagrangian method:

$$\square^2 = \square_1^2 \square_1^2 + \square_2^2 \square_1^2 \square 2 \square_1 \square_2 \square_1^2 + \square(1 \square \square_1 \square \square_2)$$
$$\frac{\partial \square^2}{\partial \square_1} = 2\square_1 \square_1^2 \square 2 \square_2 \square_1^2 \square \square$$
$$\frac{\partial \square^2}{\partial \square_2} = 2\square_2 \square_1^2 \square 2 \square_1 \square_1^2 \square \square$$

   Setting the partial derivatives equal to 0:

$$2\square_1 \square_1^2 \square 2 \square_2 \square_1^2 = 2\square_2 \square_1^2 \square 2 \square_1 \square_1^2$$
$$\square_1 \square \square_2 = \square_2 \square \square_1$$
$$2\square_1 = 2\square_2$$
$$\square_1 = \square_2$$

   Thus $\square_1 = \square_2 = 0.5$
   b. The minimum value of the portfolio variance is

$$\square^2 = 0.5^2 \square_1^2 + 0.5^2 \square_1^2 \square 2(0.5^2)\square_1^2$$
$$= 0$$

   Thus, when two assets are perfectly negatively correlated, a perfectly risk-free hedge is possible.
   c. It must be a minimum since the variance cannot be less than 0.