MPT and CAPM Mismeasure Risk

Abstract

Mean-variance analysis and the capital asset pricing model provide many useful insights for investors who want to measure and manage risk. However, their focus on short-term returns is of limited use and potentially misleading for investors with long horizons. A value investing approach suggests that risk might be better measured by long-run uncertainty about asset income than by short-run uncertainty about asset prices.

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MPT and CAPM Mismeasure Risk

Mean-variance analysis and the capital asset pricing model provide many useful insights for investors who want to measure and manage risk, including the importance of covariances among asset returns, Tobin's separation theorem, and the distinction between systematic and idiosyncratic risk. However, their focus on short-term returns is of limited use and potentially misleading for investors with long horizons. A value investing approach suggests that risk might be better measured by long-run uncertainty about asset income than by short-run uncertainty about asset prices.

Background

The early days of stock investing were characterized by little more than people guessing which direction stock prices would go next—a fertile field for bubbles and market manipulation.

In the 1930s, after having experienced the Great Crash firsthand, John Burr Williams (1938) and Benjamin Graham (1934) laid the foundation for value investing. The very first sentence of Williams's treatise, *The Theory of Investment Value*, is, "Separate and distinct things not to be confused, as every thoughtful investor knows, are real worth and market price." A stock is worth buying if the market price is less than its intrinsic value *V*, which is the present value of the future dividends, D_i, discounted by an investor's "personal rate of interest," R_i.

$$V = \frac{D_1}{\left(1+R_1\right)^1} + \frac{D_2}{\left(1+R_2\right)^2} + \frac{D_3}{\left(1+R_3\right)^3} + \ldots \qquad (1)$$

Williams recognized that dividends should be discounted by a term structure of required returns, as in Equation 1. A dividend three months from now should be discounted by a 3-month required return, a dividend 10-years hence by a 10-year required return. An investor's term structure of required returns is presumably related to the term structure of interest rates on Treasury bonds. In practice, instead of using the term structure of Treasury rates to construct a term structure of required stock returns, Williams proposed the "yield to perpetuity" on Treasury bonds: the constant interest rate that gives the same present value for an unending constant income stream as does the actual term structure of Treasury rates. The term structure of required returns in Equation 1 is then replaced by a single required return, R, that is the yield to perpetuity plus a risk premium. There is evidence that the yield to perpetuity is, in fact, a good approximation of a complete term structure (Smith and Xu 2017).

There is no need to predict future stock prices in Williams' approach because investors value stocks as if they were planning to hold them forever. Not that we literally plan to hold any stock forever, but thinking this way forces us to focus on the income from our investments instead of making speculative guesses about fluctuations in stock prices.

Williams did not explicitly consider risk beyond the recommendation that an investor's personal interest rate should be somewhat higher than the yield to perpetuity on Treasury bonds. Graham also treated risk loosely by advising investors to pay no more for a stock than a price that allows a margin of safety, "rendering unnecessary an accurate estimate of the future." Warren Buffett (1984) explained it this way:

> *You have to have the knowledge to enable you to make a very general estimate about the value of the underlying business. But you do not cut it close. That is what Ben Graham meant by having a margin of safety. You don't try to buy businesses worth $83 million for $80 million. You leave yourself an enormous margin. When you build a bridge, you insist it can carry 30,000 pounds, but you only drive 10,000 pound trucks*

across it. And that same principle works in investing.

In the 1950s Harry Markowitz (1959) and James Tobin (1958) proposed using the variances of portfolio returns to quantify investment risk—thereby creating what has come to be known as mean-variance analysis, or modern portfolio theory (MPT), which led to the Capital Asset Pricing Model (CAPM) insight that, with certain additional assumptions, beta coefficients are the proper measure of the riskiness of individual assets (Sharpe 1964, Lintner 1965, Black 1972).

In addition to providing ways to quantify risk and measure risk-adjusted returns, these models demonstrate: the importance of covariances among asset returns; Tobin's separation theorem identifying a preference-free optimal portfolio of risky assets; and the distinction between idiosyncratic risk that can be diversified away and systematic risk that cannot.

However, there are several problems with both MPT and CAPM (Fama and French 2004), including the assumption that variance is the proper measure of risk and the CAPM assumption that investors agree on the means, variances, and covariances of asset returns. It is also problematic that practitioners typically use *ex post* historical values instead of *ex ante* expectations. For example, a stock's true beta coefficient going forward generally depends on the extent to which the overall stock market will be moved by changes in the strength of the economy or by interest rates, which will almost surely not mimic what has happened in the past.

My concern is with something different. Instead of using the insights of value investing—that an investor's primary risk is uncertainty about the long-run income from an asset—MPT and CAPM are concerned with uncertainty about short-run returns, which are determined primarily by fluctuations in market prices.

For investors who do not depend on selling stock to pay their living expenses, short-run price volatility should be of little concern. If anything, fluctuations in market prices can be beneficial because they may offer opportunities for tax harvesting (Constantinides 1983, 1984, Smith and Smith 2008) and profitable transactions. If Mr. Market's prices are unreasonably high or low, value investors can take advantage of Mr. Market's foolishness. As Buffett wrote (Buffett and Loomis 2001),

> *To refer to a personal taste of mine, I'm going to buy hamburgers the rest of my life. When hamburgers go down in price, we sing the 'Hallelujah Chorus' in the Buffett household. When hamburgers go up in price, we weep. For most people, it's the same with everything in life they will be buying — except stocks. When stocks go down and you can get more for your money, people don't like them anymore.*

These models are also often misapplied to long-term valuations of stocks and real assets. When CAPM is used to determine the required return for discounting dividends from a stock or the profits from a business venture, Treasury bills and other short-term rates are almost invariably used as the risk-free asset even though Williams' yield to perpetuity is more sensible (Smith and Xu 2017). It might be argued that investors who only plan to hold a stock for a short while should use a short-term interest rate. This is analogous to arguing that investors who intend to sell a 30-year Treasury bond in three months should use the 3-month T-bill rate to discount 30 years of coupons and maturation value. This argument is clearly wrong for bonds, and it is also wrong for stocks and other long-lived assets.

Untenable Independence

The mean-variance focus on price volatility might be rationalized by the efficient-market argument that short-run price changes are independent of previous price changes and, so, investing can be viewed as a sequence of short-run investment decisions. In addition, the assumption that long-run returns are the cumulative result of a sequence of independent short-run returns simplifies the mathematical analysis enormously (for example, Tobin 1965, Samuelson 1969, Merton 1973, Bodie 1995).

Samuelson (1997) relied on the assumption of independence in this explanation of how short-run stock returns determine long-run wealth:

> *Write down those 1,800 percentage changes in monthly stock prices on as many slips of paper. Put them in a big hat. Shake vigorously. Then draw at random a new couple of thousand tickets, each time replacing the last draw and shaking vigorously. That way we can generate new realistically representative possible histories of future equity markets.*

One implication of the independence assumption is that investing is increasingly risky the longer the horizon. The appendix shows, for example, that if the gross returns are independent draws from a lognormal distribution, then the standard deviation of wealth increases with the horizon.

The problem with these analyses is that long-run movements in stock prices are not simply the cumulation of independent draws of slips of paper from a hat or numbers from a probability distribution but are, instead, constrained by fundamentals. Stock prices cannot randomly walk to permanently excessive lows or highs. This is not to say that the timing of long-run price changes can be predicted reliably but rather that, at some unknown point, price movements are affected by fundamentals. This is one reason why there is mean-reversion in stock returns (for example, Poterba and Summers, 1988; Fama and French, 1988). As George Goodman (1968), writing under the pseudonym Adam Smith, observed more than 50 years ago, "[I believe that] in the long run future earnings influence present value, and that in the short run the dominant factor is the elusive Australopithecus, the temper of the crowd."

The assumption that investing in stocks is like spinning a roulette wheel over and over has led to some erroneous conclusions and extraordinarily conservative advice. For example, Bodie (BW Online 2003) argued that,

> *Historically, stocks' standard deviation has been 20%. That means you shouldn't be at all surprised if you lose 16% rather than gain 4% in a given year. If you start out with $100,000 and lose 16%, you'll have $84,000 at the end of the first year. Then, if you lose another 16%, you'll have only $70,560 left, and so on.*

The unbounded "and so on" assumption is implausible (unless the economy is also collapsing by 16 percent a year). At some point, prices will be so low relative to corporate earnings and dividends that investors will find stocks irresistible and stock prices will stop free falling.

After the 2007-2008 crash, Bodie (Light 2009) was even more apocalyptic:

> *Prices dropped by 37% last year. While improbable, there's nothing to say they couldn't drop by that much again next year or the year before you retire. And diversification doesn't take away that risk. That's why retirement money belongs in truly safe assets whose value won't go down—not in stocks.*

Instead of considering the 37% price drop a possible buying opportunity, Bodie feared that it might be just the first of many such drops. I would argue that retirement accounts should largely ignore short-term price fluctuations—unless the investor is retired and dependent on liquidating the account.

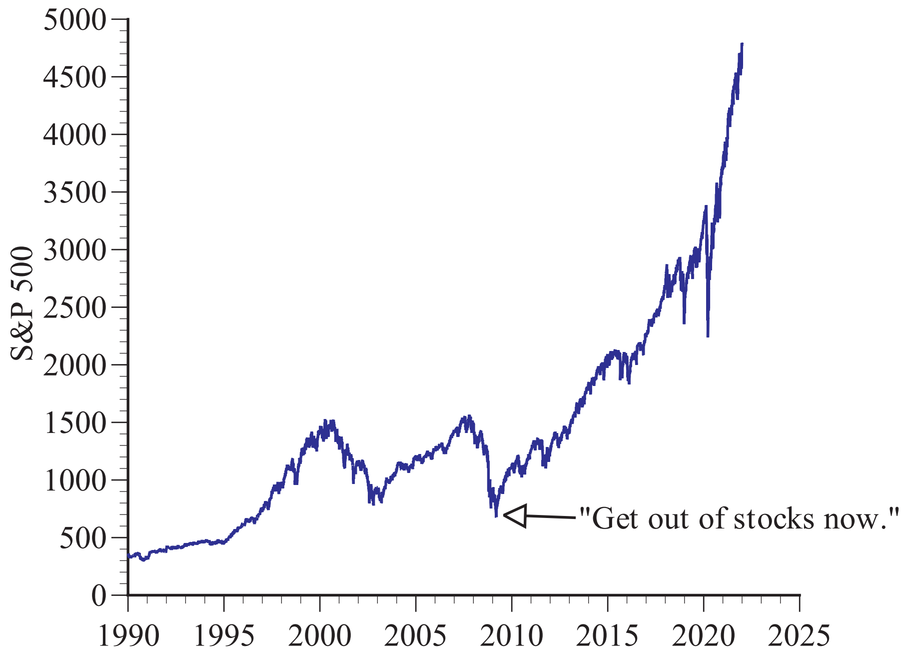
In response to the question, “So should no one invest in stocks—not even the very wealthy?,” Bodie responded as if shareholders were spinning roulette wheels and 100% losses were a realistic possible outcome: “You should only invest in equities what you can afford to lose.”

With exquisitely bad timing, in March of 2009, Bodie advised people to sell all their stocks (Blackman 2009):

> *Unless you have the heart of a high stakes gambler, get out of stocks now and put your retirement money in inflation protected government bonds and similar instruments. These investments are immune to the kind of calamity Wall Street experienced last year.*

Body’s interviewer recognized his mindset: “Bodie compares those still holding onto stocks to Las Vegas gamblers who lost big but are staying at the table trying to win it all back.” In June 2009 an interviewer (Brandon 2009) asked Bodie, “Wouldn't leaving the stock market right now be locking in your losses?,” Bodie replied, with no evident irony, “That is exactly right. You want to make sure you don’t lose more.”

Coinicdentally, the stock market bottomed on March 9, 2009, and then surged upward:



The point is not that Body was spectacularly wrong (we've all made mistakes), but that it is financially dangerous to think of the stock market as a gambling casino unhinged from the fundamental value of stocks. Stock prices will not go to zero nor will they be permanently depressed if the economy is strong. Prices will stabilize and rebound, not because of lucky draws from a hat or a probability distribution, but because stocks will be temptingly cheap if earnings and dividends rise and stock prices don't.

Four Models

The argument that short-run price volatility is a misleading measure of long-run uncertainty can be illustrated by using historical monthly S&P 500 returns back to January 1926 to calculate the mean and standard deviation of wealth for four different models over horizons ranging from 1 month to 50 years. If long-run returns are constrained by fundamental values, the volatility of the actual historical returns should be smaller than for the other models, particularly over long horizons.

Historical: Terminal wealth is calculated using each possible starting month in the database.

Shuffle: One billion Monte Carlo simulations were done with terminal wealth calculated after randomly shuffling the order of the months.

Normal: One billion Monte Carlo simulations were done assuming that each month's return is an independent draw from a normal distribution with mean and standard deviation equal to the historical values.

Lognormal: The formulas shown in the appendix were used to calculate the theoretical mean and standard deviation of terminal wealth assuming that each month's gross return is drawn from a lognormal distribution with mean and standard deviation equal to the historical values.

Results

Table 1 shows the results for 1-month, 25-year, and 50-year horizons with initial wealth set equal to 1. More complete results are shown in Figures 1 and 2.

Table 1 Mean and Standard Deviation of Wealth, Four Stock Models

	1-Month Horizon		25-year Horizon		50-Year Horizon	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
Historical	1.0097	0.054	15.78	8.41	206.49	134.53
Shuffle	1.0097	0.054	16.01	15.01	205.05	232.45
Normal	1.0097	0.054	17.37	19.77	300.03	596.89
Lognormal	1.0097	0.054	17.94	21.10	322.11	696.66

The means over long horizons are fairly close for the historical and shuffled models and for the normal and lognormal models; however, the means for the normal and lognormal models over long horizons are higher than for the historical and shuffled data. With independent draws, wealth is not constrained by fundamentals and can increase by implausible amounts.

The fact that the historical standard deviations have been substantially smaller than the shuffled-month standard deviations (and that this difference increases with the length of the horizon) supports the argument that stock returns are, in fact, anchored by fundamental values. The relatively high standard deviations for the normal and lognormal models make the point even more forcefully (and understate the standard deviations for theoretical models with fats tails). The mean and standard deviation of independent draws from a normal or lognormal distribution are a poor guide to long run means and standard deviations.

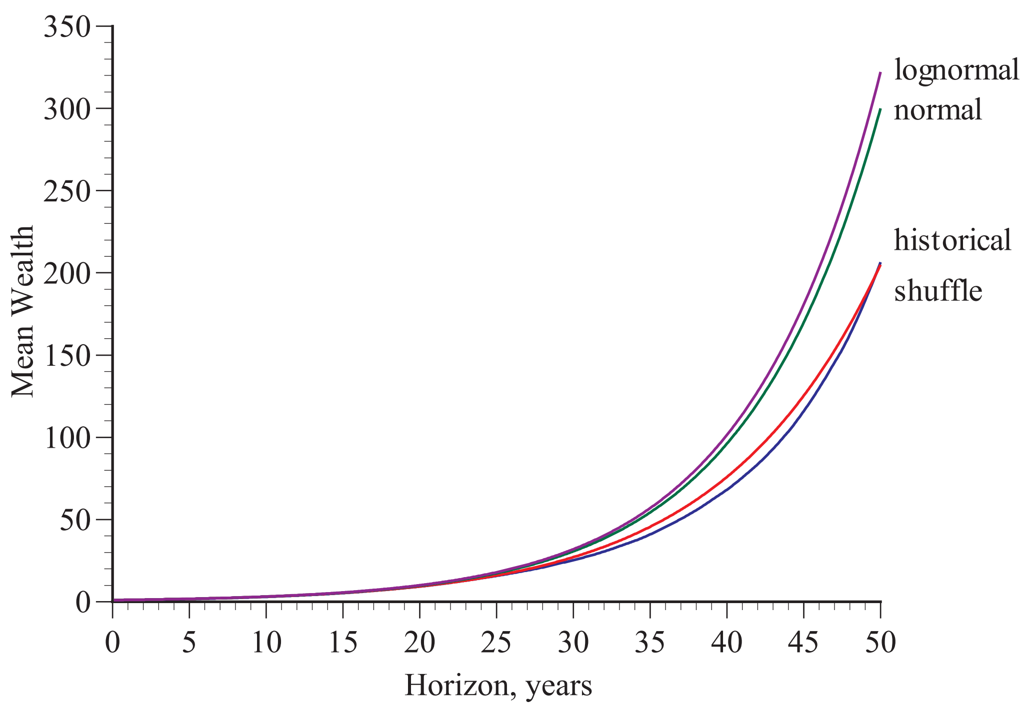


Figure 1 Mean Wealth for Four Models

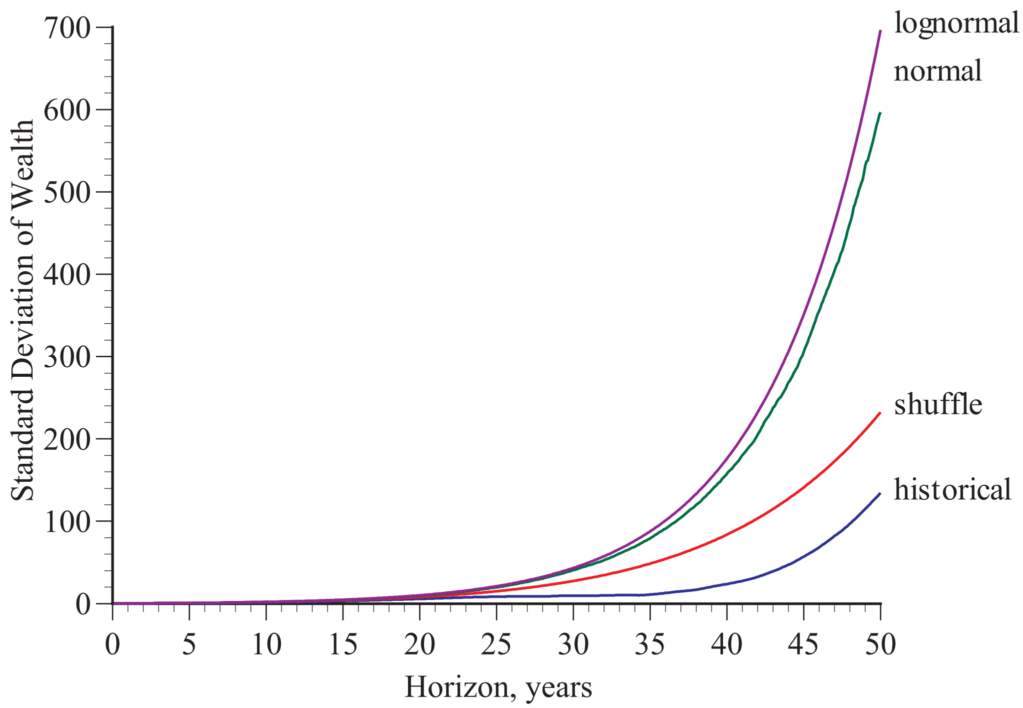


Figure 2 Standard Deviation of Wealth for Four Models

Compared to What?

Figures 1 and 2 show that both the mean and standard deviation of wealth increase with the horizon. Does the increased mean compensate for the increased standard deviation? The relevant question is how the outcomes for an all-stock strategy compare to alternative investment strategies—for example, an investment in 30-year Treasury bonds or the widely recommended 60/40 portfolio that is based on the idea that a 60% investment in stocks will give capital gains while a 40% bond investment reduces risk.

I calculated wealth, starting with $1, over horizons of up to 50 years beginning at every possible month in the 1926-2021 historical data. Table 2 shows the results for 1-month, 25-year, and 50-year horizons. The 100%-stock portfolio had a much higher average wealth and standard deviation compared to the 100%-bond portfolio, with the 60/40 portfolio in between.

Table 2 Mean and Standard Deviation of Wealth, Three Investment Strategies

	1-Month Horizon		25-year Horizon		50-Year Horizon	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
100% Stocks	1.0097	0.054	15.78	8.41	206.49	134.53
60/40	1.0077	0.035	9.89	4.68	81.49	32.46
100% Bonds	1.0046	0.028	4.66	3.06	17.48	12.00

The results summarized in Table 2 are not the most relevant statistics. The means and standard deviations would be interesting if these were independent samples but we are comparing the strategies over the same historical time period and should consider the differences in wealth; for example, how frequently the all-stock portfolio did better than the other two strategies. The answer is most of the time in the short-run and all of the time in the long run.

Table 3 shows how often an all-stock investor would have beaten the other two strategies for four different horizons. Figure 3 shows the complete results for all horizons ranging from 1 month to 50 years.

Table 3 Frequency of More Wealth With 100% Stocks, percent

Horizon	Versus 100% T-bonds	Versus 60% Stocks, 40% T-bonds
1 month	63.89	62.05
10 years	83.74	73.18
25 years	99.06	95.31
50 years	100.00	100.00

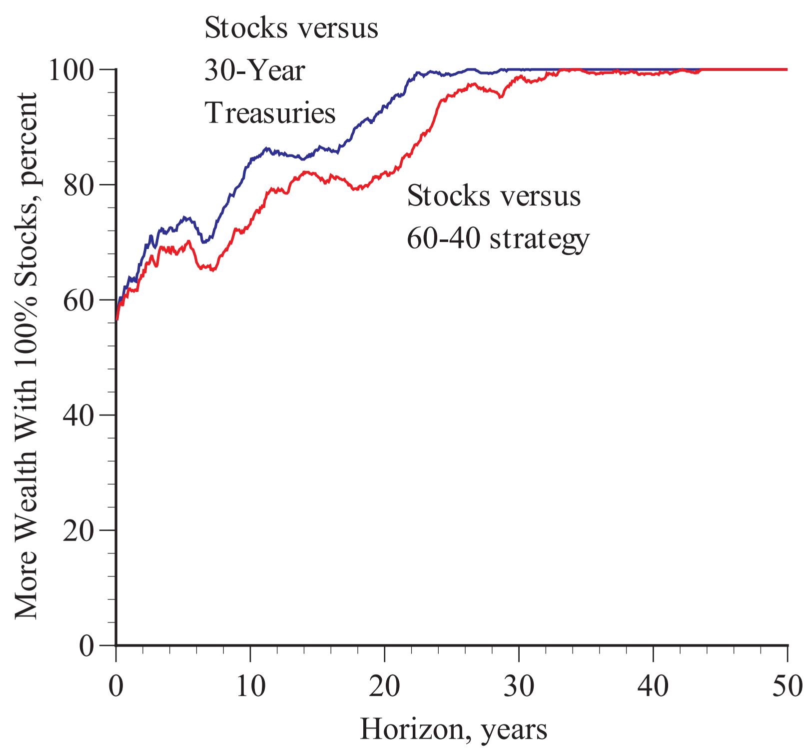


Figure 3 Frequency With Which All-Stock Portfolio Bested All-Bonds or 60/40

Table 4 shows the mean and median difference in wealth for horizons of 25 and 50 years. Figure 4 shows the distribution of the difference in wealth after 50 years for an all-stock versus 60/40 portfolio.

Table 4 Mean and Median Difference in Wealth for All-Stock Portfolio

	1-Month Horizon		25-year Horizon		50-Year Horizon	
	Mean	Median.	Mean	Median	Mean	Median
Versus 100% Bonds	0.005	0.008	11.12	7.85	189.01	145.75
Versus 60/40	0.002	0.003	5.88	3.54	125.00	91.12

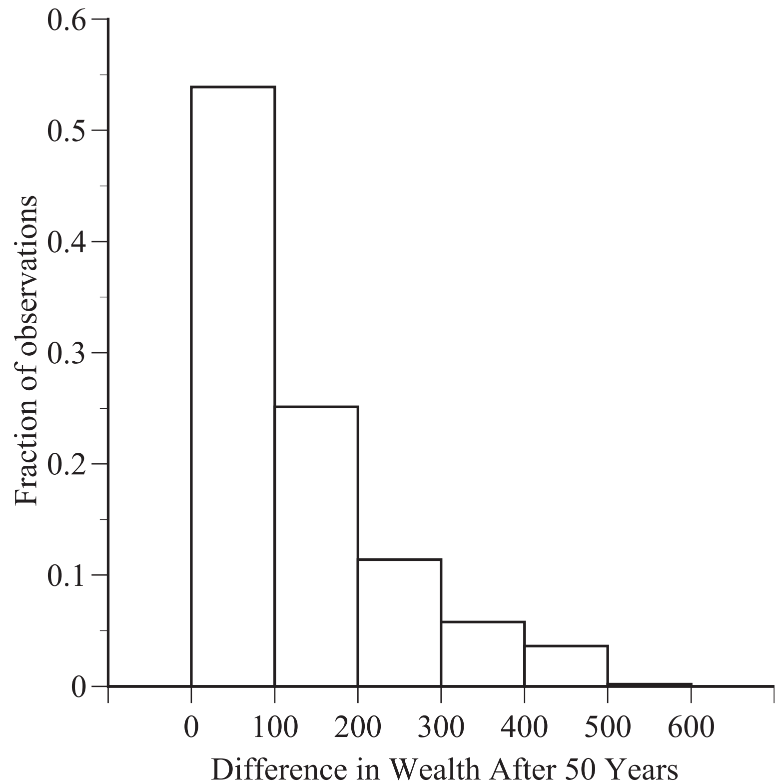


Figure 4 Wealth for 100% Stocks Minus Wealth for 60/40 Portfolio

A Better Measure of Risk

The mean-variance model is a single-period model that, in theory, could be applied to a single, very long period, say 50 years. In practice, it is not applied to long-periods because the returns over long periods tend to be highly skewed and there is plentiful evidence that skewness matters to investors (for example, Golec and Tamarkin, 1998; Garrett and Sobel, 1999; Mitton and Vorkink 2007; Kumar, 2009).

To assess risk over a long horizon, we can look again at Equation 1 with value investors discounting some measure of cash flow such as dividends, dividends plus share repurchases, free cash flow, or economic value added. The required returns are specified at the time of the valuation, taking into account the term structure of Treasury bonds. The relevant risk in gauging intrinsic value V is the uncertainty about the future cash flow.

Investors who build detailed models for projecting future cash flows can include uncertainty in the various components, either through formal probability distributions or by specifying a range of plausible scenarios with associated probabilities. For a simple example, consider the constant-dividend-growth discount model with a single required return, Williams' "personal rate of interest" based on the yield to perpetuity. Williams derived this well-known valuation equation,

$$V = \frac{D_1}{R - g} \tag{2}$$

where D_1 is the value of the next dividend (generally assumed to be one period from now), R is the investor's required rate of return, and g is the dividend growth rate. Using annual data with a \$3 dividend, 8% required return, and 5% dividend growth rate, the intrinsic value is \$100:

$$V = \frac{\$3}{0.08 - 0.05} = \$100$$

There is likely to be little uncertainty about the next dividend but considerable uncertainty about the growth rate of dividends. A probability distribution for the growth rate can be used to determine a probability distribution for the intrinsic value—in many cases, using Monte Carlo simulations.

For example, an investor might summarize his or her uncertainty about g with a truncated normal distribution:

$$g \sim N\left[0.05,\ 0.005\right] \qquad\qquad 0.03 \le g \le 0.07$$

One billion Monte Carlo simulations were used to estimate the implied probability distribution for V shown in Figure 5. This probability distribution is skewed right with a median of \$100 and mean of \$103. There is approximately a 6.7% probability that the value is less than \$80, which we can interpret by saying that, if this investor can buy the stock for \$80, he or she believes that there is only a 6.7% chance that they will have paid too much. This is a simple quantification of Graham's margin of safety.

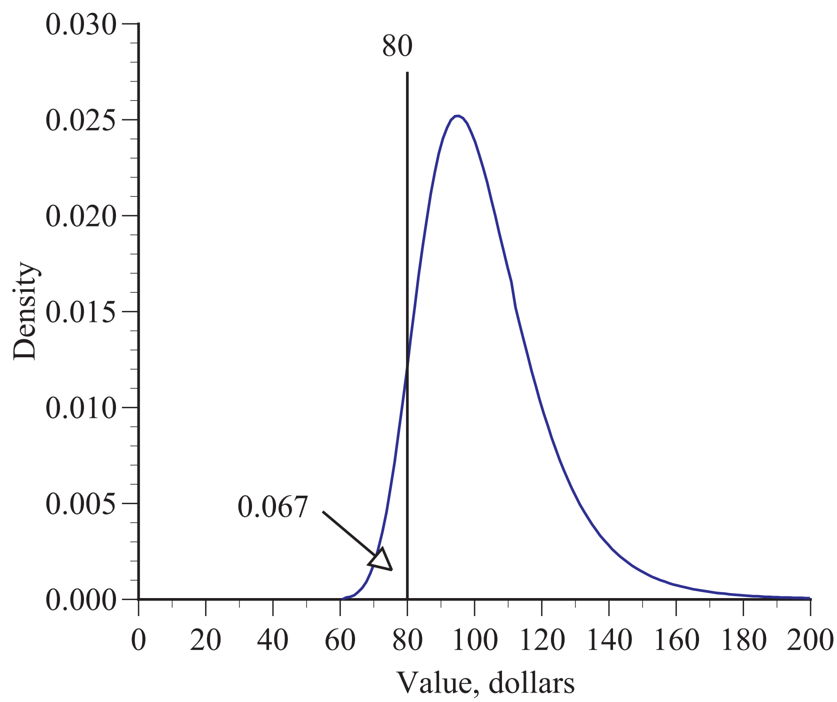


Figure 5 A Reciprocal Truncated Normal Distribution for a Stock's Intrinsic Value

Another way of presenting this probability distribution is in terms of the value surplus *VS,* the percentage difference between intrinsic value V and market price P:

$$VS = 100\left(\frac{V-P}{P}\right)$$

For a market price of \$80, there is a 0.067 probability that the value surplus is negative. The probability distribution of the value surplus shown in Figure 6 has a mean of 28.79% and a standard deviation of 23.58%.

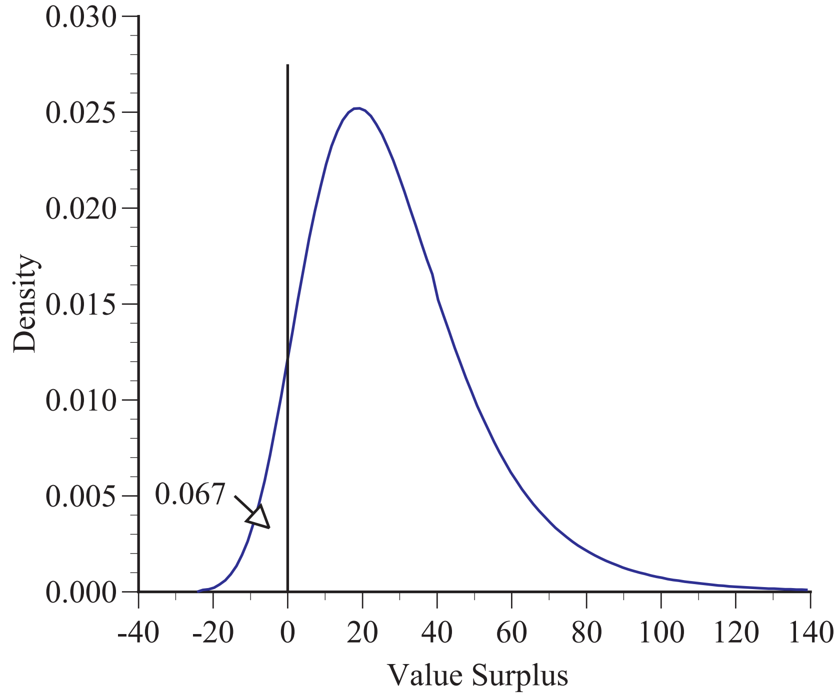


Figure 6 The Value Surplus

Alternatively, instead of a complete probability distribution, an investor might directly specify subjective probabilities for various bad-case values of *g*. For example, an investor might believe that a bad-case scenario with a 5% probability of occurring is that the dividend growth rate turns out to be only 4%, which reduces the intrinsic value to $75:

$$V = \frac{\$3}{0.06 - 0.02} = \$75$$

With this assumption, an investor who buys the stock for $75 believes that there is only a 5% chance that they overpaid.

Williams' valuation approach does not attempt to predict future wealth, either in the short run or long run, because the model makes no predictions about future stock prices. The argument is simply that buying an asset that is worth more to you than the price you paid is an attractive financial proposition.

Nonetheless, we can still demonstrate the gains from diversification by considering an investment in more than one stock. Figure 7 compares the probability distribution of the value surplus for the single asset in Figure 6 with a 50-50 investment in two assets identical to the single asset, with the investor believing that the correlation between the uncertain growth rates is zero.

This diversification does not affect the mean intrinsic value, but does reduce the spread of the probability distribution. The standard deviation of the value surplus falls from 23.58% to 16.76% and the probability that the portfolio's intrinsic value is less than the market price falls from 0.067 to 0.014.

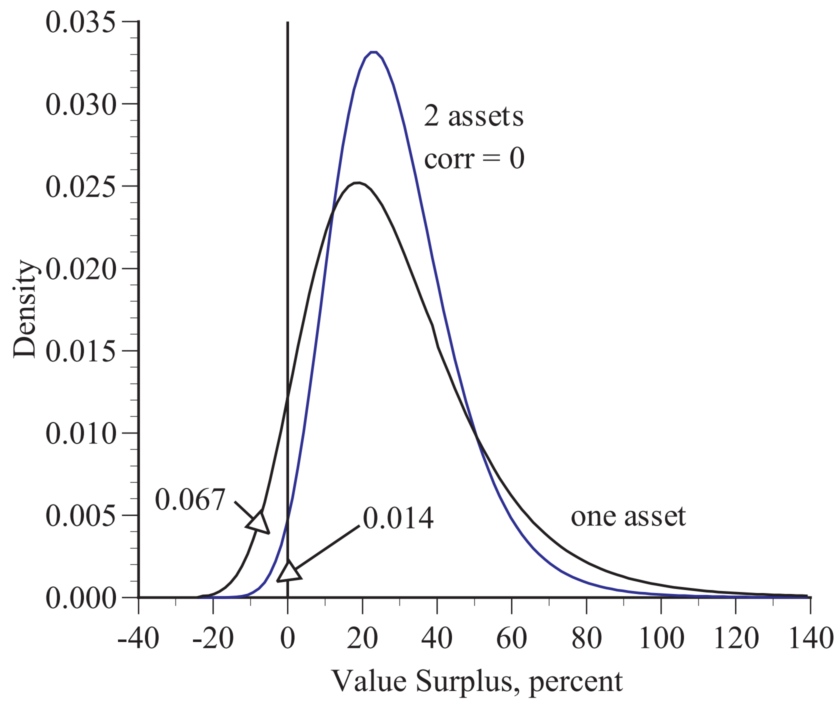


Figure 7 A 50-50 Investment in Two Stocks With Uncorrelated Growth Rates Reduces Risk

As with MPT, Figure 8 demonstrates that the amount of risk reduction provided by diversification depends on the correlation among the cash flow uncertainties, Specifically, the probability that the intrinsic value is less than the market price is smaller the lower is the correlation between the growth rates.

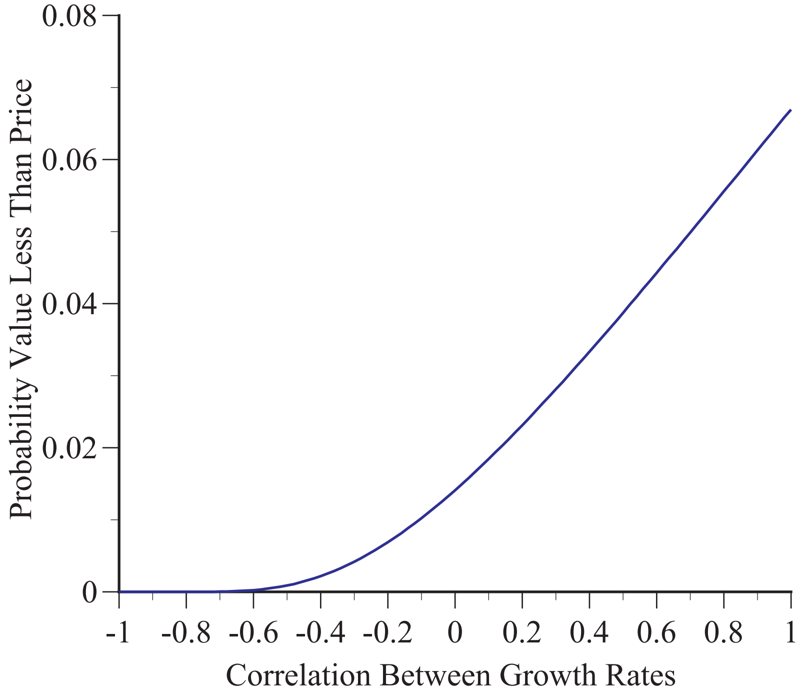


Figure 8 Diversification Reduces Risk More When Growth Rates are Less Correlated

We can also do a standard mean-variance analysis based on uncertainty about the future cash flow instead of uncertainty about price fluctuations. Figure 9 gives an example for two stocks using the constant-dividend-growth valuation in Equation 2 with truncated normal distributions for the two dividend growth rates and the parameter values in Table 5.

Table 5 Assumed Stock Parameters used in Figure 9

	Stock 1	Stock 2
D_1	3	5
Mean of g	0.05	0.03
Std. Dev. of g	0.005	0.008
Minimim g	0.03	0.01

Maximum *g*	0.07	0.05
Correlation between *g*	0.00	0.00
R	0.08	0.08
Market Price *P*	80	90
Mean of *VS*	28.79	13.87
Std. Dev. of V*S*	23.68	18.49

The MPT model gives risk-return tradeoffs for portfolio returns that depend on uncertain changes in market prices. Figure 9 shows a similar risk-return tradeoff for portfolio intrinsic value surpluses that depend on uncertain future cash flows. If there is a safe asset (presumably with no intrinsic value surplus), then Tobin's separation theorem applies, just as in the MPT model, with the optimal risky portfolio given by the tangency between a straight line from the safe asset to the opportunity locus for the risky assets.

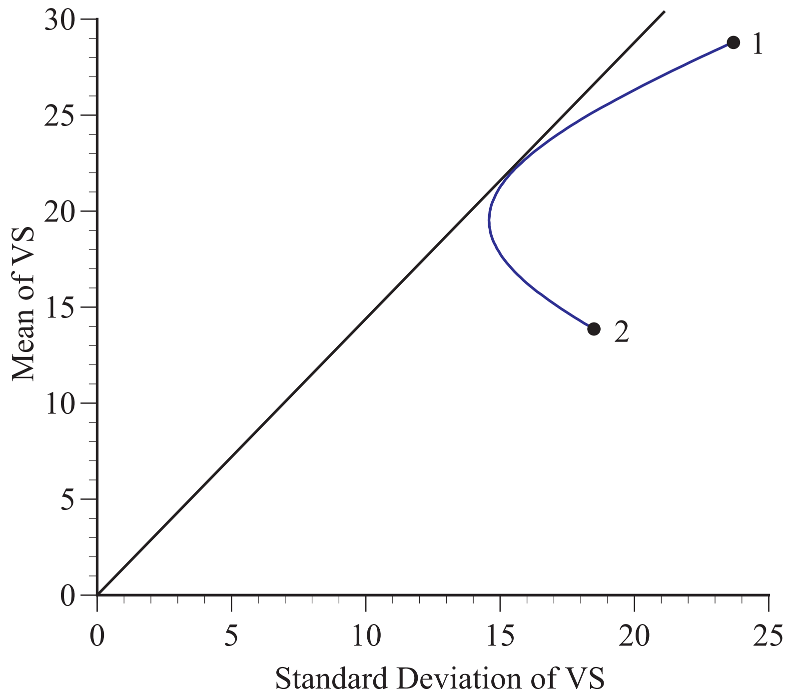


Figure 9 A Mean-Standard Deviation Tradeoff for Portfolio Value Surplus (VS)

Conclusion

MPT and CAPM both assume that the most relevant measure of risk is the variance of short-run returns, which are largely caused by the short-run volatility of market prices. An embrace of such models can lead to myopic, overly conservative strategies. For instance, the 60/40 portfolio has long been considered a prudent strategy that balances the potential price appreciation from stocks with the relative stability of bond prices. This strategy can, in fact, be an artificial drag on long-run portfolio returns because it invests too much in low-return bonds in order to reduce short-run price volatility that many investors should not care about. The your-age-in-bonds rule (40% bonds for a 40-year-old, 60% bonds for a 60-year-old) has a similar motivation and similarly unfortunate consequences.

For investors who can ignore short-run volatility in market prices (or even profit from it with tax harvesting and advantageous transactions), a more relevant uncertainty is the future cash flow from investments. A forward-looking specification of probability distributions for the cash flow can be used to determine probability distributions for intrinsic values and value surpluses. Risk might then be gauged by the probability that the value surplus is negative or by the standard deviation of the value surplus. If uncertainties about the cash flow from multiple investments are linked together through multivariate probability distributions (analogous to the covariances in MPT), then the risk associated with alternative asset portfolios can be assessed, either through the probability of a negative portfolio value surplus or by the standard deviation of the portfolio value surplus.

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Appendix Lognormal Returns Over Time

If the gross returns are independent draws from a lognormal distribution with mean *m* and variance *v*,

$$\ln\left[1+R_t\right] \sim N\left[m,v\right]$$

then the ratio of wealth at any specified horizon to initial wealth is also lognormally distributed,

$$W_n = W_o \prod_{t=1}^{n}\left(1+R_t\right)$$
$$\ln\left[\frac{W_n}{W_o}\right] = \sum_{t=1}^{n}\ln\left[1+R_t\right] \sim N\left[nm,nv\right]$$

with the mean and variance given by these equations:

$$E\left[\frac{W_n}{W_o}\right] = e^{nm+0.5nv}$$
$$\mathrm{var}\left[\frac{W_n}{W_o}\right] = \left(e^{nv}-1\right)e^{2nm+nv}$$