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THE EFFECT OF POPULATION GROWTH ON WEALTH AND SAVING IN THE MODIGLIANI–BRUMBERG LIFE CYCLE MODEL

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The effect of a change in the rate of population growth on the aggregate savings–income ratio and on the aggregate wealth–income ratio is evaluated in a life cycle model of capital accumulation and growth. It is shown that the former will increase and the latter will decrease when the rate of population growth increases.

1. Introduction

In their classic 1953 paper Modigliani and Brumberg described a now familiar life cycle model of saving behavior. The continuing appeal of this model is due to its quite remarkable blend of a simple analysis with very reasonable conclusions.

One area in which the analysis does become complicated and yet should be of great interest concerns the aggregate implications of changes in the rate of population growth. There is a hint in the 1953 paper which Modigliani (1966) states much more directly that an increase in the rate of population growth will raise the aggregate savings–income ratio and lower the wealth–income ratio. Yet in both papers this informed speculation is not formally proven but instead supported by a few illustrative calculations.

The present note proves that Modigliani's conjecture is indeed correct. These proofs relate to a discrete period model since this is the framework used in the original (1953) article. The proofs for the analogous continuous model are quite similar.

2. The model

There is no uncertainty. Each individual begins life with only human wealth and leaves no bequests. Each person lives for n periods and will receive a constant labor income y per period for the first $m(<n)$ of these periods. Interest rates and the rate of time preference are zero, so that each individual spreads out human wealth so as to consume a constant amount each period

$$c = my/n\,. \tag{1}$$

At the beginning of the ith period of one's life accumulated financial wealth is given by

$$w_i = \begin{cases} (i-1)\dfrac{n-m}{n}\,y\,, & 1 \leqslant i \leqslant m+1\,, \\ (n-i+1)\dfrac{m}{n}\,y\,, & m+1 \leqslant i \leqslant n+1\,. \end{cases} \tag{2}$$

For an aggregate analysis, assume a smoothly growing population with p_i individuals in the ith year of a life cycle

$$p_i = p_{i-1}/(1+g) = p_1\left(\frac{1}{1+g}\right)^{i-1}, \qquad 1 \leqslant i \leqslant n, \qquad g > -1. \tag{3}$$

Using (1), (2), and (3) aggregate income, consumption, and wealth can be written as

$$Y = \sum_{i=1}^{m} y p_i = y p_1 \sum_{i=1}^{m} b^{i-1}\,,$$

$$C = \sum_{i=1}^{n} c p_i = \frac{m}{n} y p_1 \sum_{i=1}^{n} b^{i-1}\,,$$

$$W = \sum_{i=1}^{n} w_i p_i = y p_1 \left[\sum_{i=1}^{m} (i-1) b^{i-1} - \frac{m}{n}\sum_{i=1}^{n} (i-1) b^{i-1} + m \sum_{i=m+1}^{n} b^{i-1}\right], \tag{4}$$

where $b = 1/(1+g) > 0$. Since each age group is expanding at the rate g the aggregate population, income, consumption, and wealth are also growing at this rate. And since the growth of wealth is equal to aggregate saving,

$$S \equiv Y - C = W_{+1} - W = gW\,,$$

and each individual has nonnegative wealth, it follows that aggregate saving will have the same sign as the rate of population growth. I will assume $g > 0$. Using (4), an upper bound can be placed on the savings–income ratio by

$$\begin{aligned} nC - mY &= myp_1 \sum_{1}^{n} b^{i-1} - myp_1 \sum_{1}^{m} b^{i-1} \\ &= myp_1 \sum_{m+1}^{n} b^{i-1} \geqslant 0\,. \end{aligned}$$

Thus for a nonnegative rate of population growth,

$$\frac{m}{n} \leqslant \frac{C}{Y} \leqslant 1 \quad \text{and} \quad 0 \leqslant \frac{S}{Y} \leqslant 1 - \frac{m}{n}\,. \tag{5}$$

3. Population growth and the savings–income ratio

Using (4) the steady state implications for S/Y of a faster rate of population growth can be obtained by differentiating

$$\frac{C}{Y}=\frac{m}{n}\left(\sum_{0}^{n-1} b^i\right)/\sum_{0}^{m-1} b^i ,$$

with respect to b. This gives

$$n\left(\sum_{0}^{m-1} b^i\right)^2 \frac{\partial(C/Y)}{\partial b} = \left(\sum_{0}^{m-1} mb^i\right)\left(\sum_{1}^{n-1} ib^{i-1}\right) - \left(\sum_{0}^{n-1} mb^i\right)\left(\sum_{1}^{m-1} ib^{i-1}\right)$$

$$= \left(\sum_{0}^{m-1} mb^{i-1}\right)\left(\sum_{m}^{n-1} ib^i\right) - \left(\sum_{1}^{m-1} ib^{i-1}\right)\left(\sum_{m}^{n-1} mb^i\right) > 0 , \quad b>0,$$

so that

$$\frac{\partial(S/Y)}{\partial g} > 0 , \qquad g>-1. \tag{6}$$

Each worker is saving $(n-m)/(n)y$ while each retired person is dissaving $(m)/(n)y$. Each worker raises the aggregate saving–income ratio while each retiree lowers it; m workers will have aggregate saving just equal to the aggregate dissaving of $n-m$ retirees. Since there are m age brackets for workers and $n-m$ for the retired, aggregate saving will be zero if there are the same number of people in each age bracket. The faster the rate of growth of population the more the age distribution shifts toward workers and the greater is aggregate saving and the saving–income ratio.

4. Population growth and the wealth–income ratio

The wealth–income ratio is given by

$$\frac{W}{Y} = \left[\sum_{1}^{m-1} ib^i + m\sum_{m}^{n-1} b^i - \frac{m}{n}\sum_{1}^{n-1} ib^i\right]/\sum_{0}^{m-1} b^i . \tag{7}$$

Differentiating (7) with respect to b gives

$$b\left(\sum_{0}^{m-1} b^i\right)^2 \frac{\partial(W/Y)}{\partial b} = \sum_{0}^{m-1} b^i\left[\sum_{1}^{m-1} i^2b^i + m\sum_{m}^{n-1} ib^i - \frac{m}{n}\sum_{1}^{n-1} i^2b^i\right]$$

$$- \sum_{1}^{m-1} ib^i\left[\sum_{1}^{m-1} ib^i + m\sum_{m}^{n-1} b^i - \frac{m}{n}\sum_{1}^{n-1} ib^i\right]$$

$$= \sum_0^{m-1} b^i \left[\left(1 - \frac{m}{n}\right) \sum_1^{m-1} i^2 b^i + \frac{m}{n} \sum_m^{n-1} i(n-i)\, b^i\right]$$

$$- \sum_1^{m-1} ib^i \left[\left(1 - \frac{m}{n}\right) \sum_1^{m-1} ib^i + \frac{m}{n} \sum_m^{n-1} (n-i)\, b^i\right]$$

$$= \left(1 - \frac{m}{n}\right) [\sum_0^{m-1} b^i \sum_1^{m-1} i^2 b^i - \sum_1^{m-1} ib^i \sum_1^{m-1} ib^i]$$

$$+ \frac{1}{n} [\sum_0^{m-1} mb^i \sum_m^{n-1} i(n-i)\, b^i - \sum_1^{m-1} ib^i \sum_m^{n-1} m(n-i)\, b^i]$$

$$= \left(1 - \frac{m}{n}\right) Z_1 + \frac{1}{n} Z_2 \; .$$

Z_2 can be directly seen to be positive. At $m = 1, Z_1 = 0$. Z_1 is positive for any $m > 1$ since at any positive value of m, an increase in m by 1 unit increases Z_1 by

$$b^m \sum_1^{m-1} i^2 b^i + m^2 b^m \sum_0^{m-1} b^i - mb^m \sum_1^{m-1} ib^i - mb^m \sum_1^{m-1} ib^i$$

$$= b^m \sum_0^{m-1} (i^2 + m^2 - 2im)\, b^i$$

$$= b^m \sum_0^{m-1} (m-i)^2 b^i > 0 \; .$$

For the intuition of this, the aggregate wealth–income ratio can be rewritten as the summation of the age bracket ratio of wealth to worker income, weighted by each age group's population relative to the aggregate working population

$$\frac{W}{Y} = \sum_{i=1}^{n} \frac{w_i}{y} \, (p_i / \sum_{i=1}^{m} p_i) \, .$$

An increase in the rate of population growth reduces the age group weights,

$$p_i / \sum_{i=1}^{m} p_i \, ,$$

for all retirees. For workers ($i \leqslant m$), w_i/y rises with age and an increase in the rate of population growth tilts the age group weights so as to raise

$$p_i / \sum_{i=1}^{m} p_i \, ,$$

for younger workers and lower it for older workers. Thus, for both workers and retirees an increase in the rate of population growth tends to reduce the aggregate wealth–income ratio.

References

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